

# On unconditional Banach space Ideal Property

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## Abstract

Let  $Lw'$  denote the assignment which associates with each pair of Banach spaces  $X, Y$  the vector space  $Lw'(X, Y)$  and  $K(X, Y)$  be the space of all compact linear operators from  $X$  to  $Y$ . Let  $T \in Lw'(X, Y)$  and suppose  $(T_n) \subset K(X, Y)$  converges in the dual weak operator topology ( $w'$ ) of  $T$ . Denote by  $Ku((T_n))$  the finite number given by

$$Ku((T_n)) := \sup_{n \in \mathbb{N}} \{ \max \{ \|T_n\|, \|T - 2T_n\| \} \}.$$

The  $u$ -norm on  $Lw'(X, Y)$  is then given by

$$\|T\|_u := \inf \{ Ku((T_n)) : T = w' - \lim_n T_n, \quad T_n \in K(X, Y) \}.$$

It has been shown that  $(Lw'(X, Y) \|\cdot\|_u)$  is a Banach operator ideal. We find conditions for  $K(X, Y)$  to be an unconditional ideal in  $(Lw'(X, Y) \|\cdot\|_u)$ .

**Key words:** *Ideal projection, separable reflexive space, unconditional compact approximation property (UKAP),  $u$ -ideal.*

## 1. Introduction

In section 8 of paper [2], the authors established necessary conditions on a Banach space  $X$  such that the space  $K(X)$  of compact operators is a  $u$ -ideal in the space  $L(X)$  of bounded linear operators, showing that this is the case if  $X$  is separable and has (UKAP) (unconditional compact approximation property, i.e. if there exists a sequence  $(K_n)$  in  $K(X)$  such that

$$\lim K_n x = x \text{ for all } x \in X \text{ and } \lim_n \|id_X - 2K_n\| = 1).$$

Johnson proved in [5] that if  $Y$  is a Banach space having the bounded approximation property then the annihilator  $K(X, Y)^\perp$  in the (continuous) dual space  $L(X, Y)^*$  is the kernel of a projection on  $L(X, Y)^*$ . The range space of the projection is isomorphic to the dual space  $K(X, Y)^*$ . K. John showed in [3] that Johnson's result is also true in case of any separable Pisier space  $X = P$  and its

dual  $Y = P^*$ , both being spaces which do not have the approximation property. This motivated his more general results in a later paper,(cf [4]).

In the paper [1] an alternative (operator ideal) approach is followed to prove similar (and more general) versions of John's results. Having proved that  $(L^{w'}, \|\cdot\|_u)$  is a Banach operator ideal (cf [6]), we shall build on the results in [1] to obtain conditions for the space  $K(X, Y)$  of compact operators to be a  $u$ -ideal in a suitable subspace  $(L^{w'}(X, Y), \|\cdot\|_u)$  of  $L(X, Y)$ . If  $L^{w'}(X, Y) = L(X, Y)$ , our results states conditions on  $L(X, Y)$  so that  $K(X, Y)$  is a  $u$ -ideal in  $L(X, Y)$ .

Before we investigate the  $u$ -ideal property of  $K(X, Y)$  in  $(L^{w'}(X, Y), \|\cdot\|_u)$ , we recall from [2] the ideal property of  $K(X, Y)$  in  $L^{w'}(X, Y)$  with respect to the  $\|\cdot\|_w$ -norm.

**Theorem 1.1:** (cf. [1], Theorem 2.5) *There exists a continuous bilinear form*

$J : L_{w'}(X, Y)^* \times L_{w'}(X, Y) \rightarrow \mathbf{K}$  *such that*

- (a)  $J(\phi, T) = \phi(T)$  *for all*  $(\phi, T) \in L_{w'}(X, Y)^* \times L_{w'}(X, Y)$
- (b)  $|J(\phi, T)| \leq \|\phi\| \|\phi(T)\|$  *for all*  $T \in L_{w'}(X, Y)$  *and*  $\phi \in L_{w'}(X, Y)^*$ .
- (c)  $J(\phi, T) = \lim_n \phi(T_n)$ , *where*  $(T_n)$  *is any sequence of compact operators*  $T_n \in K(X, Y)$  *tending to*  $T$  *in*  $w$ -*topology.*

*In particular,*

**Corollary 3.1.2:** *Let*  $X, Y$  *be Banach spaces. There is a projection*

$$P : (L_{w'}(X, Y), \|\cdot\|_w)^* \rightarrow (L_{w'}(X, Y), \|\cdot\|_w)^*$$

*such that*

$$\text{Ker}(P) = K(X, Y)^\perp = \{\phi \in L_{w'}(X, Y)^* : \phi(K(X, Y)) = 0\}, \|P\| \leq 1$$

*and the range of*  $P$  *is isomorphic to*  $K(X, Y)^*$ . *The projection*  $P$  *is given by*

$$P\phi(T) = \lim_n \phi(T_n) = J(\phi, T). \text{ Thus } K(X, Y) \text{ is an ideal in } (L_{w'}(X, Y), \|\cdot\|_w).$$

*We shall make use of the properties of a Banach ideal operator defined in terms of compact approximation properties and separable reflexive spaces.*

## 2. Unconditional Ideal property.

The authors in [2] call a sequence  $(K_n)$  of compact operators from  $X$  into  $X$  a *compact approximation sequence* if  $\lim_n K_n x = x$  for all  $x \in X$ . In [2] it is also agreed to say that  $X$  has (UKAP) if there is a compact approximation sequence

$$K_n : X \rightarrow X \text{ such that } \lim_{n \rightarrow \infty} \|I - 2K_n\| = 1.$$

It is also proved in [2] that if  $X$  is a separable Banach space, then  $X$  has (UKAP) if and only if for every  $\epsilon > 0$  there is a sequence  $(A_n)$  of compact operators such that for every  $x \in X$  and every  $n$  and every  $\theta_j = \pm 1$ ,  $1 \leq j \leq n$  we have  $\sum_{i=1}^n A_i x = x$  and  $\|\sum_{i=1}^n \theta_j A_j x\| \leq (1 + \epsilon) \|x\|$ .

In particular, this means that if we let  $K_n = \sum_{i=1}^n A_i$ , then  $K_n x \rightarrow x$ ,  $\forall x \in X$  and

$$\|K_n x\| \leq (1 + \epsilon) \|x\|, \quad \forall x \in X, \quad \forall n \in \mathbf{N}.$$

Moreover, also  $\|I - 2K_n\| \leq 1 + \epsilon$ .

When a separable Banach space  $X$  has  $UKAP$ , it is easily seen that for each  $T \in L(X)$ ,  $TK_n \rightarrow T$  (as  $n \rightarrow \infty$ ) is the weak operator topology. If  $X$  is also reflexive, then  $TK_n \rightarrow T$  (as  $n \rightarrow \infty$ ) in the  $w'$ -topology and it follows that

$$Ku((TK_n)) \leq (1 + \epsilon) \|T\|.$$

Since  $\epsilon > 0$  is arbitrary, it follows that  $\|T\|_u \leq \|T\|$ , i.e.  $\|T\| = \|T\|_u$  in this case.

Putting  $T_n := TK_n$ , it follows that  $T_n \xrightarrow{w'} T$  and  $\|T - 2T_n\| \leq (1 + \epsilon) \|T\|$  for all  $n \in \mathbf{N}$ . Consequently, it follows that

$$\begin{aligned} \|Id_{(Lw')^*} - 2P\| &= \sup_{\|\phi\| \leq 1} \|\phi - 2P\phi\| \\ &= \sup_{\|\phi\| \leq 1} \sup_{\|\phi\| \leq 1} |\phi(T) - 2P\phi(T)| \\ &= \sup_{\|\phi\| \leq 1} \sup_{\|\phi\| \leq 1} |\lim_n \phi(T - 2T_n)| \\ &\leq \sup_{\|T\| \leq 1} \sup_n \|T - 2T_n\| \leq 1 + \epsilon \end{aligned}$$

This being so for all  $\epsilon > 0$ , it is clear that:

**Proposition 2.1:** (Special case of [2], Proposition 8.2). Let  $X$  be a separable reflexive Banach space. If  $X$  has  $(UKAP)$ , then  $K(X)$  is a  $u$ -ideal in  $L(X)$ .

If  $X$  satisfies the conditions in Proposition 2.1 and  $Y$  is any Banach space, then for each  $T \in L(X, Y)$  and each  $\epsilon > 0$ , we may choose the sequence  $(K_n) \subset K(X)$  to satisfy the properties in the above proof of Proposition 2.1.

Again, put  $T_n = TK_n$  for all  $n$ . Then, as before,  $T_n \xrightarrow{w'} T$  and we still have the inequalities

$$\|T - 2T_n\| \leq (1 + \epsilon) \|T\| \text{ and } Ku((T_n)) \leq (1 + \epsilon) \|T\|.$$

Hence  $\|T\|_u \leq (1 + \epsilon) \|T\|$  for all  $\epsilon > 0$ . The existence of a contractive projection  $P : L(X, Y)^* \rightarrow L(X, Y)^*$  with  $\text{Ker}(P) = K(X, Y)^\perp$  follows from the Theorem 1.1 and Corollary 1.2, since in this case we have  $(L(X, Y), \|\cdot\|) = (Lw'(X, Y), \|\cdot\|_u)$ .

Therefore,  $K(X, Y)$  is an ideal in  $L(X, Y)$ .

The argument in the proof of Proposition 2.1 then shows that:

**Proposition 2.2:** Let  $X$  be a separable reflexive Banach space and  $Y$  any Banach space. If  $X$  has  $(UKAP)$ , then  $K(X, Y)$  is a  $u$ -ideal in  $L(X, Y)$ .

In the discussion of the proof of Proposition 2.1 it is important to realise that for each

$T \in L(X, Y)$  and each  $\epsilon > 0$  the sequence  $(T_n) \subset K(X, Y)$  can be chosen to satisfy  $T_n \xrightarrow{w'} T$  and  $\|T - 2T_n\| \leq (1 + \epsilon) \|T\|$  and  $\|T_n\| \leq (1 + \epsilon) \|T\|$ .

With the conditions on the Banach space  $X$  in Proposition 2.1, the norms  $\|\cdot\|$ ,  $\|\cdot\|_u$  and  $\|\cdot\|_u$  coincide on  $L(X, Y)$ , exactly because we can choose the sequence  $(T_n)$  as such. Therefore, it is natural to formulate the following definition:

**Definition 2.3:** Let  $X$  and  $Y$  be Banach spaces. We say an operator  $T \in L(X, Y)$  has  $(w' - UKAP)$  (i.e. it has the “ $w'$ -uniform compact approximation property” if each  $\epsilon > 0$  there exists a sequence  $(T_n) \subset K(X, Y)$  such that

$$\|T - \lim_n T_n\| \leq (1 + \epsilon) \|T\| \text{ and } \|T_n\| \leq (1 + \epsilon) \|T\| \text{ for all } n.$$

It follows from the above discussion that:

**Proposition 2.4:** Suppose each  $T \in (L_{w'}(X, Y), \|\cdot\|_u)$  (respectively, each  $T \in L(X, Y)$ ) has  $(w' - UKAP)$ . Then  $K(X, Y)$  is a  $u$ -ideal in  $(L_{w'}(X, Y), \|\cdot\|_u)$  (respectively, each  $T \in L(X, Y)$ ).

We follow the prove of proposition 2.1 which was already discussed in [2] that a separable reflexive Banach space has  $(UKAP)$  if and only if  $K(X)$  is a  $u$ -ideal in  $L(X)$ .

As a motivation for the condition  $(w' - UKAP)$  in Proposition 2.4, we introduce another property on Banach spaces:

A sequence  $(K_n)$  of compact operators from  $X$  into  $X$  is called a  $w'$ -compact approximating sequence if  $w' - \lim_n K_n = I$ .

If  $X$  is reflexive, then clearly each compact approximating sequence is  $w'$ -compact approximating. We say  $X$  has  $(w' - UKAP)$  if for each  $\epsilon > 0$  there is a  $w'$ -compact approximating sequence  $K_n : X \rightarrow X$  such that  $\|K_n x\| \leq (1 + \epsilon) \|x\| \quad \forall x \in X, \forall n \in \mathbb{N}$  and  $\|I - 2K_n\| \leq 1 + \epsilon$  for all  $n$ .

Finally from proposition 2.4 we formulate the following corollary:

**Corollary 2.5:** If  $X$  has  $(w' - UKAP)$ , then  $K(X)$  is a  $u$ -ideal in  $L(X)$ .

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